

# A Generalization of a Result of Hardy and Littlewood

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## Abstract

In this note we study the growth of

$$\sum_{m=1}^M \frac{1}{\|m\alpha\|}$$

as a function of  $M$  for different classes of  $\alpha \in [0, 1)$ . Hardy and Littlewood showed in [2] that for numbers of bounded type, the sum is  $\simeq M \log M$ . We give a very simple proof for it. Further we show the following for generic  $\alpha$ . For a non-decreasing function  $\varphi$  tending to infinity,

$$\limsup_{M \rightarrow \infty} \frac{1}{\varphi(\log M)} \left[ \frac{1}{M \log M} \sum_{m=1}^M \frac{1}{\|m\alpha\|} \right]$$

is zero or infinity according as

$$\sum \frac{1}{k\varphi(k)}$$

converges or diverges.

## 1. INTRODUCTION

In 1920's and 30's, Hardy and Littlewood made significant progress in the area of Diophantine approximation. They often relied on advanced techniques: deep results from complex analysis, Cesàro summability,  $L$ -functions, etc. Among the simpler tools were continued fractions; main results in this field had been

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## PRELIMINARY RESULTS

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established by Gauss and Legendre and appeared in a complete form, for example, in [1] (first edition published in 1889). In spite of this fact the new approach presented here is based entirely on the theory of continued fractions. We will show explicitly how the elements of the continued fraction expansion of  $\alpha$  govern the behavior of the sum<sup>1</sup>

$$\sum_{m=1}^M \frac{1}{\|m\alpha\|}. \quad (1.1)$$

The primary source of facts on continued fractions is [8]; we will quote results from this book and direct the reader to it for proofs. We will however present proofs of theorems from [8] for which we have more elementary proofs.

We start by stating the following instrumental

**Lemma 1.1.** *Take any irrational  $\alpha \in [0, 1)$  and an integer  $k > 1$ . Let  $\frac{p_k}{q_k}$  be the  $k$ -th convergent to  $\alpha$ . Then, we have the inequality*

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}. \quad (1.2)$$

This Lemma [8, theorems 9 and 3] shows how well any given number can be approximated by its convergents. It will be key in estimating our sums as the “largest” terms in our sums will typically appear when  $m = q_k$  for some  $k$ .

First we shall tackle the lower bound for the sum

$$S_M(\alpha) = \sum_{m=1}^M \frac{1}{\|m\alpha\|}.$$

Then, we shall derive an upper bound. Whenever we write  $S_M(\alpha)$ , we imply that it is defined; i.e.,  $\alpha \notin \mathbf{Q}$ .

## 2. PRELIMINARY RESULTS

Let us introduce some useful terminology and basic tools. A real number is said to be of *bounded type* if the elements of its continued fraction expansion are bounded. It is shown in [8] and follows from Theorem 3.2 that the measure of bounded type numbers is zero.

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<sup>1</sup>In what follows we use the notation  $\|x\| = \min_{n \in \mathbf{Z}} |x - n|$ . We use  $f = \mathcal{O}(g)$  and  $f \ll g$  interchangeably and if  $f \gg g \gg f$ , we write  $f \simeq g$ .

**Tool 2.1.** *For real numbers  $x, y$ , we have*

$$\|x + y\| \leq \|x\| + \|y\|.$$

*Proof.* Let  $\|x\| = |x - u|$  and  $\|y\| = |y - v|$  for integers  $u$  and  $v$ . Then, by the triangle inequality for real numbers,

$$\|x\| + \|y\| = |x - u| + |y - v| \geq |x + y - (u + v)| \geq \min_{n \in \mathbf{Z}} |x + y - n| = \|x + y\|.$$

□

**Lemma 2.2.** *For each  $m \in \{1, \dots, q_{k+1}\}$ , we have*

$$\left\| \frac{mp_k}{q_k} \right\| - \frac{1}{q_k} < \|m\alpha\| < \left\| \frac{mp_k}{q_k} \right\| + \frac{1}{q_k}. \quad (2.1)$$

for  $k > 1$ .

*Proof.* We make use of the right-hand side of (1.2). Multiplying it by a positive integer  $m \in \{1, \dots, q_{k+1}\}$  gives

$$\left| m\alpha - \frac{mp_k}{q_k} \right| < \frac{m}{q_k q_{k+1}} \leq \frac{1}{q_k}.$$

Since  $q_k \geq 2$ , we have  $\frac{m}{q_k q_{k+1}} \leq \frac{1}{2}$ , and thus

$$\left\| m\alpha - \frac{mp_k}{q_k} \right\| = \left| m\alpha - \frac{mp_k}{q_k} \right| < \frac{1}{q_k}. \quad (2.2)$$

Applying Tool 2.1 to the above inequality in two different ways we get the desired result. □

### 3. LOWER BOUND ON $S_M(\alpha)$

Now we have enough instruments to begin analyzing the first sum. Theorems 3.1 and 4.1 will give respectively lower and upper bounds for the sum.

**Theorem 3.1.** *For any  $\alpha \in [0, 1)$ , fix  $k$  so that  $q_k \leq M < q_{k+1}$ . We have*

$$S_M(\alpha) \gg M \log q_k,$$

with an absolute implied constant.

*Proof.* We use the right-hand side of (2.1) to approximate our sum. We shall then use the fact that  $(p_k, q_k) = 1$  to simplify the bound.

We sum from  $m = 1 + lq_k$  to  $(l+1)q_k$  for some  $l$  such that  $(l+1)q_k \leq M$ :

$$\sum_{m=1+lq_k}^{(l+1)q_k} \frac{1}{\|m\alpha\|} \geq \sum_{m=1+lq_k}^{(l+1)q_k} \frac{1}{\left\| \frac{mp_k}{q_k} \right\| + \frac{1}{q_k}} \gg q_k \sum_{m=1+lq_k}^{(l+1)q_k} \frac{1}{[mp_k] + 1} = q_k \sum_{m=1}^{q_k} \frac{1}{[mp_k] + 1}.$$

The notation  $[t]$  means the integer  $t' \in \{0, \dots, q_k - 1\}$  such that  $t \equiv t' \pmod{q_k}$ . As  $(p_k, q_k) = 1$ , the latter expression requires that we sum reciprocals of integers from 1 to  $q_k$ . The sum on the right becomes

$$\sum_{n=1}^{q_k} \frac{1}{n} \gg \log q_k.$$

So we have

$$\sum_{m=1+lq_k}^{(l+1)q_k} \frac{1}{\|m\alpha\|} \gg q_k \log q_k.$$

The number of intervals  $[1 + lq_k, (l+1)q_k]$  over which we summed is  $\simeq \frac{M}{q_k}$ , whence

$$\frac{M}{q_k} q_k \log q_k = M \log q_k.$$

□

To further analyze the lower bound we invoke the following theorem of Khinchin:

**Theorem 3.2** (Khinchin). *Let  $l$  stand for Lebesgue measure on  $[0, 1]$  and let  $a_k: [0, 1] \rightarrow \mathbf{N}$  denote the  $k$ -th entry of the continued fraction expansion. For any  $\varphi: \mathbf{N} \rightarrow \mathbf{R}^+$ ,*

$$l\{a_k > \varphi(k) \text{ i.o.}\} = 1 \text{ or } 0$$

according as

$$\sum_k \frac{1}{\varphi(k)} \text{ is divergent or convergent.}$$

One direction in the proof is a direct application of the Borel-Cantelli Lemma as it doesn't rely on independence, while the other requires some more work. Details that exhibit sufficient independence among the elements of continued fractions as well as the proof in its entirety can be found in [8].

Next, we discuss two lemmata on metric theory of continued fractions; we don't truly capitalize on them until we get to the upper bound of  $S_M(\alpha)$ . The proof of the first Lemma can be found in [7, 5] and in other introductory literature on ergodic theory. The second Lemma is proven in [8] but we use the proof from [6] which is nicer and elucidates the ergodic nature of the Gauss map.

**Lemma 3.3** (Invariant measure for the Gauss map). *The transformation*

$$\begin{aligned} T: ([0, 1), \mathcal{B}[0, 1), l) &\rightarrow ([0, 1), \mathcal{B}[0, 1), l) \\ x &\mapsto \frac{1}{x} - \left[ \frac{1}{x} \right], \end{aligned}$$

where  $\mathcal{B}$  stands for the Borel  $\sigma$ -field and  $l$  is Lebesgue measure on it, is ergodic and its invariant measure is

$$d\mu(x) = \frac{1}{\log 2} \cdot \frac{dl(x)}{x+1}.$$

**Lemma 3.4** (Exponential growth of partial quotients). *There exist positive constants  $a$  and  $A$  such that the statement  $a < \sqrt[k]{q_k} < A$  holds a.e. on  $[0, 1)$  for  $k$  sufficiently large.*

**Remark.** The lower bound is actually true for all  $k$  and  $\alpha$ . It suffices to notice that  $q_k$  grow at the lowest rate when  $\alpha = [1, 1, 1, \dots] = \frac{\sqrt{5}-1}{2}$ . In this case,  $q_k$  are consecutive Fibonacci numbers and grow geometrically with common ratio  $\frac{\sqrt{5}+1}{2} > 1$ . The second statement is false on a non-empty negligible set. Take  $\alpha = [1, 2, 2^2, \dots]$ . Then,  $a_1 a_2 \dots a_k = 2^{\frac{k(k-1)}{2}}$ . So,  $(a_1 \dots a_k)^{1/k} = 2^{\frac{k-1}{2}} \rightarrow \infty$  as  $k \rightarrow \infty$ . It follows from the proof below that for this  $\alpha$  the sequence  $\sqrt[k]{q_k}$  is unbounded, too. The theorem is probably due to Khinchin, but we present a simpler proof found in [6]. Khinchin strengthened the conclusion to  $\sqrt[k]{q_k} \rightarrow \gamma$  a.s. in [3], and Lévy showed in [4] that  $\log \gamma = \frac{\pi^2}{12 \log 2}$ .

*Proof.* Let  $T$  and  $\mu$  be as in Lemma 3.3. Then for any  $\alpha \in [0, 1)$  we have that  $a_k(\alpha) = \left[ \frac{1}{T^{k-1}\alpha} \right]$ . It is plain that

$$a_k q_{k-1} < a_k q_{k-1} + q_{k-2} = q_k < a_k q_{k-1} + q_{k-1} \leqslant 2a_k q_{k-1}.$$

Proceeding inductively we conclude that  $a_1 \dots a_k < q_k < 2^k a_1 \dots a_k$ . Hence it suffices to show that

$$\sqrt[k]{\prod_{i=1}^k a_i} \rightarrow \text{const} > 0 \quad \text{a.s.}$$

Here we invoke Lemma 3.3. Take  $f(x) = \log \left[ \frac{1}{x} \right] \in L^1$ . By the Pointwise Ergodic Theorem,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \left[ \frac{1}{T^i x} \right] = \frac{1}{\log 2} \int_0^1 \log \left[ \frac{1}{x} \right] \frac{dx}{1+x} = C, \quad \text{a.s., rate depending on } \alpha$$

It follows by exponentiation that

$$e^C = \lim_{k \rightarrow \infty} (a_1 \dots a_k)^{1/k}, \quad \text{a.s.}$$

and our proof is complete.  $\square$

We now apply these facts to the case at hand. For numbers of bounded type, we have  $M < q_{k+1} = a_{k+1}q_k + q_{k-1} < Cq_k$  and thus we can claim that

$$S_M(\alpha) \gg M \log M$$

with a universal implied constant. If  $\alpha$  is of unbounded type, then it is quite possible that growth is slower than  $M \log M$ . We will prove this later by calculating an upper bound that is lower than  $M \log M$  for very ill-behaved  $\alpha$ . Almost surely, however, we do get the  $M \log M$  lower bound. By Theorem 3.2, the set  $\{a_{k+1} \leq q_k \text{ eventually}\}$  is of full measure. Indeed,  $q_k$  grow at least geometrically everywhere and hence the complement  $\{a_{k+1} > q_k \text{ i.o.}\}$  is null. Thus, on the event  $\{a_{k+1} \leq q_k \text{ eventually}\}$ ,

$$\log M \leq \log(a_{k+1}q_k + q_{k-1}) \ll \log q_k^2 \ll \log q_k$$

and up to a constant,  $\log M$  and  $\log q_k$  are the same.

To summarize,  $S_M(\alpha) \gg M \log M$  for  $\alpha$  of bounded type and for almost all  $\alpha$ . That this estimate fails to hold on the entire interval  $[0, 1)$  will be demonstrated in the next section.

#### 4. UPPER BOUND ON $S_M(\alpha)$

Now we proceed to obtain the upper bound. This result will not be uniform in  $\alpha$ : specifically, if  $\alpha$  is of unbounded type, then our conclusions are weaker.

**Theorem 4.1.** *Let the continued fraction expansion of  $\alpha \in [0, 1)$  be  $[a_1, a_2, \dots]$ . Given an integer  $M$ , take  $k$  so that  $q_k \leq M < q_{k+1}$ . Then we have*

$$S_M(\alpha) \ll M \log q_k + a_{k+1}M.$$

*Proof.* Let's begin by applying the left inequality of (2.1). There are three cases when we don't gain any information from this inequality: when  $[mp_k] = 0, \pm 1$ . We exclude these for now as they require a different treatment. Summing over the remaining  $m$  from 1 to  $q_k$  we get

$$\sum_{\substack{1 \leq m \leq q_k \\ [mp_k] \neq 0, \pm 1}} \frac{1}{\|m\alpha\|} \leq \sum_{\substack{1 \leq m \leq q_k \\ [mp_k] \neq 0, \pm 1}} \frac{1}{\left\| \frac{mp_k}{q_k} \right\| - \frac{1}{q_k}} \ll q_k \sum_{\substack{1 \leq m \leq q_k \\ [mp_k] \neq 0, \pm 1}} \frac{1}{[mp_k] - 1}.$$

As in the proof of Theorem 3.1, we notice that the factor of  $p_k$  is superfluous as  $(p_k, q_k) = 1$ . Thus, the sum can rewritten as

$$q_k \sum_{\substack{1 \leq n \leq q_k \\ [n] \neq 0, \pm 1}} \frac{1}{n - 1}.$$

Clearly, this quantity is asymptotic to  $q_k \log q_k$ . There will be  $\simeq \frac{M}{q_k}$  such terms, inasmuch as we get the first term in the bound.

Now we deal with terms  $mp_k \equiv 0, \pm 1$ . We shall estimate the error incurred by replacing  $\|m\alpha\|$  by  $\|q_k\alpha\|$ . I claim that  $\|m\alpha\| \geq \frac{1}{2}\|q_k\alpha\|$  for any  $m$  such that  $q_k \leq m < q_{k+1}$ . This is certainly true if  $m = q_k$ , so suppose  $m$  is *not* a partial quotient of  $\alpha$ . Proceed by contradiction. Assume that  $\|m\alpha\| < \frac{1}{2}\|q_k\alpha\|$ . By (1.2), it follows that  $\|q_k\alpha\| < \frac{1}{q_{k+1}}$ . Thus,

$$\|m\alpha\| < \frac{1}{2}\|q_k\alpha\| < \frac{1}{2q_{k+1}} < \frac{1}{2m}.$$

By Theorem 19 in [8], we get that  $m$  is a partial quotient of  $\alpha$ , which contradicts our assumption.

Hence at the expense of a factor of two, we can replace  $\|m\alpha\|$  by  $\|q_{k+1}\alpha\|$ . Thus, the bound for the sum

$$\sum_{[mp_k]=0, \pm 1} \frac{1}{\|m\alpha\|} \tag{4.1}$$

is  $\frac{M}{q_k} \frac{1}{\|q_k\alpha\|} \ll a_{k+1} M$ . Indeed, from the left-hand side of (1.2) it follows that  $\|q_k\alpha\| \gg \frac{1}{q_{k+1}}$ , and desired result follows at once.  $\square$

One may wonder whether it is possible to improve on the second term or to get rid of it altogether. Indeed, we have nonchalantly replaced the sum by the

product of the number of terms and the largest term. In general, the answer is no. If the sum consists of but one term, little can be done to improve the approximation  $\|q_k \alpha\| \approx \frac{1}{q_{k+1}}$  as can be seen from (1.2). On the other hand, if the sum consists of many terms (which is the same as saying that  $M$  is close to the upper end of the interval), we can improve the bound to  $M(1 + \log a_{k+1})$ . This improvement is based entirely on careful analysis of (4.1) and is left to the reader.

We can now discuss some of the consequences of this Theorem. First of all, we have established a result of [2] using a shorter and more elementary method. Clearly, for numbers of bounded type we have the upper bound  $M \log M$ , and the implied constant depends on  $\alpha$  in both cases. It is curious that in this case the main contribution comes from the “bulk” terms (with  $\frac{1}{\|m\alpha\|}$  small), while the “special” terms contribute less.

We also address now the issue from the previous section. Namely, we prove existence of numbers for which the statement

$$S_M(\alpha) \gg M \log M$$

does not hold. Suppose all  $\alpha$  satisfy this relation. Then, by Theorem 4.1,

$$M \log q_k + a_{k+1}M \gg S_M(\alpha) \gg M \log M.$$

It suffices to show that the estimate fails for one subsequence of  $M$ . So, we pick  $M \sim q_{k+1}$ . Thus, we have

$$\log q_k + a_{k+1} \geq C q_k a_{k+1}$$

for some fixed positive  $C$ . For  $k$  large enough,  $C q_k$  will exceed unity. Consider the lines  $y = \log q_k + x$  and  $y = C_1 x$  in the  $xy$ -plane with  $C_1 > 1$ . It is plain that for  $x$  large enough, the second line will lie above the first and thus  $a_{k+1}$  can be picked in violation of the above inequality. All subsequent  $a_j$  can be picked in the same way and thus our assumption is false: it is not necessarily true that  $S_M(\alpha) \gg M \log M$ . Loosely speaking, these  $\alpha$  have sufficiently rapidly growing  $a_j$  and thus form a set of measure zero, as required by discussion following Theorem 3.1.

## 5. GROWTH CRITERION

We have an upper bound and a lower bound for  $S_M(\alpha)$ . What is the function that captures the exact growth rate of the sum? The following Theorem lets one decide whether the sum grows faster or slower than any given function.

**Theorem 5.1.** *Let  $\varphi(x)$  be a positive non-decreasing function. Then, for almost every  $\alpha \in [0, 1)$  we have*

$$\limsup_{M \rightarrow \infty} \frac{S_M(\alpha)}{M \log M} \Big/ \varphi(\log M) = \begin{cases} 0 & \text{if } \sum_k \frac{1}{k\varphi(k)} < \infty \\ \infty & \text{if } \sum_k \frac{1}{k\varphi(k)} = \infty. \end{cases}$$

*Proof.* Let's verify the first line. By Theorem 4.1 we have

$$\frac{S_M(\alpha)}{M \log M \varphi(\log M)} \ll \frac{1}{\varphi(\log M)} + \frac{a_{k+1}}{\log M \varphi(\log M)} \ll \frac{1}{\varphi(k \log a)} + \frac{a_{k+1}}{k \varphi(k \log a)}$$

where  $a$  comes from Theorem 3.4. Now by Khinchin's Theorem,

$$l\{a_{k+1} > k\varphi(k \log a) (\varepsilon - \frac{1}{\varphi(k \log a)}) \text{ i.o.}\} = 0$$

for any positive  $\varepsilon$  since

$$\sum_{k=1}^{\infty} \frac{1}{k\varphi(k \log a)(\varepsilon - \frac{1}{\varphi(k \log a)})} \simeq \sum_{k=1}^{\infty} \frac{1}{k\varphi(k \log a)} \simeq \int \frac{dx}{x\varphi(x)} < \infty.$$

We remark here that the sum extends over those  $k$  for which  $\varepsilon - \frac{1}{\varphi(k \log a)} > 0$  and that  $\log a > 0$  by direct computation. Hence the quantity

$$\frac{S_M(\alpha)}{M \log M} \Big/ \varphi(\log M)$$

exceeds the value  $\varepsilon$  finitely often with probability one. Taking  $\varepsilon = \frac{1}{n}$  for  $n \in \mathbf{N}$  we get a countable union of measure zero sets, which itself has measure zero, inasmuch as the limit superior vanishes almost surely, as advertized.

We go on to prove the second line. We shall concentrate on the term  $m = q_k$  and exhibit a suitable sequence  $M_k$  so that this term of the sum alone will diverge as  $M \rightarrow \infty$ . It is natural to take  $M_k = q_k$ . Then, we need to estimate

$$\frac{1}{\|q_k \alpha\| q_k \log q_k \varphi(\log q_k)}.$$

Using (1.2), we get using Lemma 3.4 that it is greater than

$$\frac{q_{k+1}}{q_k \log q_k \varphi(\log q_k)} \gg \frac{a_{k+1}}{\log q_k \varphi(\log q_k)} \gg \frac{a_{k+1}}{k \varphi(k \log A)}$$

## CONCLUSION

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for  $k$  sufficiently large. Thus, by Khinchin's Theorem, we need to ensure that  $\sum \frac{1}{k\varphi(k \log A)} = \infty$ . This is indeed the case since  $\varphi$  is non-decreasing and the sum can be compared to the integral. Hence, the quantity in question is almost surely unbounded.  $\square$

## 6. CONCLUSION

The above discussion summarizes completely the behavior of  $S_M(\alpha)$  for all  $M$ . Theorem 5.1 gives the exact growth of the sum for almost all  $\alpha$ .

We have obtained results for some measure zero sets, too. For numbers of bounded type,

$$\frac{S_M(\alpha)}{M \log M} \simeq 1.$$

We can in fact say more. Tracing the source of constants in Theorems 3.1 and 4.1, we observe that the constants arise from changing

$$\sum_m \frac{1}{\|x_m\| \pm 1} \quad \text{to} \quad \sum_m \frac{1}{\{x_m\} \pm 1}.$$

Both changes introduce a constant factor of two. Therefore, the strongest statement is that

$$\frac{S_M(\alpha)}{M \log M} \rightarrow f(\alpha)$$

has a limiting distribution. Furthermore, for any given number with rapidly growing partial quotients we can establish a specific bound.

It is worth pointing out that all the results above can be easily adapted to the sum

$$\sum_{m=1}^M \frac{1}{\|m\alpha\|^\beta}$$

for  $\beta > 1$ . Other expressions can be allowed in the sum, too. However, summation may become difficult as the simplification that we have used in Theorem 3.1 might not work. For example,

$$\sum_{m=1}^M \frac{1}{m\|m\alpha\|}$$

is harder to estimate, and more general summations like this one are a direction of further research.

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## REFERENCES

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To better understand the growth of the original quantity  $S_M(\alpha)$ , it may be advantageous to look at averages of the sum:

$$\frac{1}{N} \sum_{n=1}^N S_n(\alpha).$$

This is so because we have seen that for certain values of  $M$  the sum is unusually large, while for others it is quite small. The general behavior could be elucidated through Cesàro means of this kind. This is possible direction for future investigation.

## References

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